Trajectories of semigroups of holomorphic functions and harmonic measure

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Abstract

We give an explicit relation between the slope of the trajectory of a semigroup of holomorphic functions and the harmonic measure of the associated planar domain Ω. We use this to construct a semigroup whose slope is an arbitrary interval in $[-\pi/2,\ \pi/2]$. The same method is used for the slope of a backward trajectory approaching a super-repulsive fixed point.

Keywords: semigroups of holomorphic functions, harmonic measure, trajectories, slope

2010 MSC: 30C20, 30C85, 30D05

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1. Semigroups of Holomorphic Functions

A one-parameter continuous semigroup of holomorphic self-mappings of the unit disk $\mathbb{D}$ is a family $(\phi_t)_{t\in[0,\infty)}$, such that:

(i) $\phi_{t+s} = \phi_t \circ \phi_s$, for all $t, s \in [0, +\infty)$

(ii) $\phi_0(z) = z$

(iii) $\lim_{t\to s} \phi_t(z) = \phi_s(z)$, for all $s \in [0, +\infty)$.

We will simply call (ϕ_t) a semigroup. For general reference on semigroups we point to [1], [12] and [16].

A semigroup is called *elliptic* if it is not a group of hyperbolic rotations and it has an interior fixed point, which must be the same for all $\phi_t,\ t > 0$. If (ϕ_t) is a non-elliptic semigroup, then there exists a unique point $\xi \in \partial\mathbb{D}$, called the Denjoy-Wolff point of the semigroup [2], such that

$$\lim_{t\to\infty} \phi_t(z) = \xi, \quad \text{for every } z \in \mathbb{D}. \tag{1}$$

A semigroup with no interior fixed point is called *non-elliptic*. From now on we will only deal with non-elliptic semigroups. An important tool in the study of non-elliptic semigroups is the corresponding Koenigs function, see [1], [12], [16] and the references therein. To every non-elliptic semigroup (ϕ_t), corresponds a conformal mapping $h : \mathbb{D} \to \Omega$ such that:

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(i) $h(\mathbb{D}) = \Omega$, and

(ii) $h(\phi_t(z)) = h(z) + t,\ z \in \mathbb{D},\ t \geq 0$.

The domain Ω is called the associated planar domain of (ϕ_t). A domain Ω is called convex in the positive direction when $\{z + t : z \in \Omega\} \subset \Omega$, for all $t \in [0, \infty)$. Obviously the associated planar domain of a semigroup is convex in the positive direction. The converse is also true; for every simply connected domain Ω convex in the positive direction, define

$$\phi_t(z) = h^{-1}(h(z) + t),$$

where h is the Riemann map that maps $\mathbb{D}$ onto Ω. It is easy to verify that (ϕ_t), as defined above, is a semigroup.

We are interested in the boundary fixed points of ϕ_t. These are defined using the notion of angular limit. When $\phi(z) \to w'$ as $z \to w$ through any sector at w we say that w' is the angular limit of ϕ as z tends to w; we write

$$\angle \lim_{z \to w} \phi(z) = w'.$$

A point $w \in \partial\mathbb{D}$ is called a boundary fixed point of ϕ, when $\angle \lim_{z \to w} \phi(z) = w$. For a boundary fixed point w, we define the angular derivative at w to be

$$\phi'(w) = \angle \lim_{z \to w} \frac{w - \phi(z)}{w - z}.$$

In the case when $\phi(\mathbb{D}) \subset \mathbb{D}$, we know [14, p.82] that $\phi'(w)$ always exists and belongs to $(0, +\infty) \cup \{\infty\}$. Boundary fixed points in this case are divided into three categories; see [8] and references therein.

(i) When $\phi'(w) \in (0, 1]$, w is called an attractive point,

(ii) when $\phi'(w) \in (1, +\infty)$, w is called a repulsive point and

(iii) when $\phi'(w) = \infty$, w is called a super-repulsive point.

The Denjoy-Wolff Theorem guarantees that, in the context of semigroups, the Denjoy-Wolff point ξ in relation (1), is the unique attractive boundary fixed point of ϕ_t, for all $t > 0$.

Non-elliptic semigroups can be categorized according to properties of the associated planar domain Ω; see e.g. [3]. Namely:

(i) When Ω is contained in a horizontal strip, the semigroup is called hyperbolic.

(ii) When Ω is not contained in a horizontal strip, but it is contained in a horizontal half-plane, the semigroup is called parabolic of positive hyperbolic step.

(iii) When Ω is not contained in any horizontal half-plane, the semigroup is called parabolic of zero hyperbolic step.

The trajectory of $z \in \mathbb{D}$ of a semigroup (ϕ_t) is defined as the curve

$$\gamma_z : [0, +\infty) \to \mathbb{D},\ \gamma_z(t) = \phi_t(z).$$

By utilizing the associated domain Ω, every trajectory can be extended as follows. Let T be the infinum of $\{t : h(z) + t \in \Omega\}$. The extended trajectory of z is the curve defined by

$$\gamma_z : (T, +\infty) \to \mathbb{D},\ \gamma_z(t) = h^{-1}(h(z) + t). \tag{2}$$

From now on γ_z will be used for the extended trajectory. In accordance with [8], we will define the α and ω limits of curves. For every curve $\Gamma : (s_1, s_2) \to \mathbb{C}$, if there exists a strictly increasing sequence $t_n \to s_2$, such that $\Gamma(t_n) \to \xi$, then ξ is called an ω-limit point of Γ. The set of all ω-limit points of Γ is called the ω-limit set and denoted by $\omega(\Gamma)$. Replacing s_2 with s_1 and considering strictly decreasing sequences, we similarly define the α-limit point and the α-limit set $\alpha(\Gamma)$. From (1) it is obvious that for all $z \in \mathbb{D}$ we have $\omega(\gamma_z) = \{\xi\}$, where ξ is the Denjoy-Wolff point. The set $\alpha(\gamma_z)$ is also a single point which can be one of the following [8]:

(i) The point in $\partial\mathbb{D}$ that corresponds to $h(z) + T \in \partial\Omega$, when $T > -\infty$.

(ii) A boundary fixed point of (ϕ_t), including the Denjoy-Wolff point ξ, when $T = -\infty$.

An interesting problem is the study of the slope of γ_z as it approaches the boundary of $\mathbb{D}$. For every γ_z, we consider the corresponding curve

$$t \in (T, +\infty) \to \arg(1 - \bar{\xi}\gamma_z(t)) \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{3}$$

The ω-limit set of the above curve will be the set of slopes of γ_z as it approaches the Denjoy-Wolff point ξ and it will be denoted by $\texttt{Slope}^+(\gamma_z)$. If $\alpha(\gamma_z) = \{\chi\}$ then similarly consider the curve

$$t \in (T, +\infty) \to \arg(1 - \bar{\chi}\gamma_z(t)) \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{4}$$

The α-limit set of the above curve will be called the set of slopes of the backward trajectory γ_z as it approaches the boundary point χ and it will be denoted by $\texttt{Slope}^-(\gamma_z)$. The following is already known about the $\texttt{Slope}^+(\gamma_z)$.

(i) When a semigroup is hyperbolic, $\texttt{Slope}^+(\gamma_z)$ is a singleton depending on z.

(ii) When a semigroup is parabolic of positive hyperbolic step, $\texttt{Slope}^+(\gamma_z)$ is either $\{\pi/2\}$ or $\{-\pi/2\}$ and it is independent of z.

When a semigroup is parabolic of zero hyperbolic step, it was conjectured that $\texttt{Slope}^+(\gamma_z)$ is again a singleton. This was proven but only under some additional assumptions, see e.g. [10] and [11]. The existence of a semigroup with $\texttt{Slope}^+(\gamma_z) = [-\pi/2, \pi/2]$ was first proven in [4] and [9]. In a more recent result, Bracci et al. [5] show that there exists a semigroup such that $\texttt{Slope}^+(\gamma_z) \subset (-\pi/2, \pi/2)$ but it is not a singleton. Also in [6] we find an example with $\texttt{Slope}^+(\gamma_z) = [-\pi/2, \alpha]$, for some $-\pi/2 < \alpha < \pi/2$.

In [9] the authors posed the problem of constructing examples of one-parameter semigroups (ϕ_t) with $\texttt{Slope}^+(\gamma_z) = [\theta_1, \theta_2]$ for any given $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$, $\theta_1 < \theta_2$. We will construct such a semigroup.

Theorem 1. *If $\theta_1 < \theta_2$ are real numbers with $|\theta_j| \leq \pi/2$, $j = 1, 2$, then there exists a semigroup of holomorphic functions (ϕ_t) such that*

$$\texttt{Slope}^+(\gamma_z) = [\theta_1, \theta_2]. \tag{5}$$

For the $\texttt{Slope}^-(\gamma_z)$ similar results were only known for the following cases [8]:

(i) When the α-limit of γ_z is the Denjoy-Wolff point ξ, $\texttt{Slope}^-(\gamma_z)$ is a singleton, which is either $\{\pi/2\}$ or $\{-\pi/2\}$.

(ii) When the α-limit of γ_z is a repulsive point, $\texttt{Slope}^-(\gamma_z)$ is a single point, which belongs in $(-\pi/2, \pi/2)$.

We prove that, in the case of super-repulsive points, a semigroup can have a wildly oscillating trajectory, quite similar to the case of a parabolic semigroup of zero hyperbolic step.

Theorem 2. *If $\theta_1 \leq \theta_2$ are real numbers with $|\theta_j| \leq \pi/2$, $j = 1, 2$, then there exists a semigroup of holomorphic functions (ϕ_t) and a point $z \in \mathbb{D}$, such that the α-limit of γ_z is a super-repulsive point and*

$$\texttt{Slope}^-(\gamma_z) = [\theta_1, \theta_2].$$

2. Harmonic measure

To prove the aforementioned results we need to establish a relationship between the slope of a trajectory γ_z and certain harmonic measures in the associated planar domain Ω of a semigroup.

The harmonic measure is the solution u of the generalized Dirichlet problem for the Laplacian in a domain D, with boundary values equal to 1 on $E \subset \partial\Omega$ and 0 on $\partial\Omega \setminus E$. We will be using the notation $\omega(z, E, D)$.

Two basic properties of the harmonic measure that we will use are conformal invariance and domain monotonicity. When $\phi : \mathbb{D} \to \Omega$ is a conformal map, we know that, if A is the set of accessible points of $\partial\Omega$, we can extend ϕ^{-1} to A. In that sense, when $E \subset A$ we have [13, p.206]

$$\omega(z, \phi^{-1}(E), \mathbb{D}) = \omega(\phi(z), E, \Omega). \tag{6}$$

This implies that when an arc $\widehat{ab} \subset \partial\mathbb{D}$ corresponds, through ϕ, to a boundary set $E \subset \partial\Omega$, in the sense of Caratheodory boundary correspondence, then

$$\omega(z, \widehat{ab}, \mathbb{D}) = \omega(\phi(z), E, \Omega). \tag{7}$$

When for two domains D_1, D_2 in $\mathbb{C}_\infty$, with $D_1 \subset D_2$, we have a set $B \subset \partial D_1 \cap \partial D_2$, then [15, p. 102]

$$\omega(z, B, D_1) \leq \omega(z, B, D_2). \tag{8}$$

We also know that [7, p.155], if $\widehat{ab} \subset \partial\mathbb{D}$ is a circular arc, then the level set

$$L_k = \{\zeta \in \mathbb{D} : \omega(\zeta, \widehat{ab}, \mathbb{D}) = k\},\ 0 < k < 1, \tag{9}$$

is a circular arc with endpoints a and b that meets the unit circle with angle $k\pi$. We will also use the notation

$$\widehat{L}_k = \{\zeta \in \mathbb{D} : \omega(\zeta, \widehat{ab}, \mathbb{D}) > k\}. \tag{10}$$

In order to establish a relation between certain harmonic measures in the case when D contains, in a specific way, a rectangle, we introduce the following notation.

For any set B in the complex plane $\mathbb{C}$, let $B^+ = B \cap \{z : \operatorname{Im} z > 0\}$ and $B^- = B \cap \{z : \operatorname{Im} z < 0\}$. Let

$$S_d = \{z : -d < \operatorname{Im} z < d\} \tag{11}$$

be a horizontal strip of width $2d$,

$$S_{d,u} = \{z : -d < \operatorname{Im} z < d,\ \ -u < \operatorname{Re} z < u\} \tag{12}$$

be a rectangle centered at the origin with width $2d$ and length $2u$,

$$B_{d,u} = \{z : \operatorname{Im} z = d,\ \ -u < \operatorname{Re} z < u\} \tag{13}$$

be the upper side of $S_{d,u}$ and $B_{-d,u}$ be the lower side of $S_{d,u}$. Betsakos [4] has proven the following:

Lemma 1. *Let Ω be a planar domain, convex in the positive direction. Assume that $\mathbb{R} \subset \Omega$ and that $(\partial\Omega)^+ \neq \emptyset$, $(\partial\Omega)^- \neq \emptyset$. Let $\epsilon > 0$ and $d > 0$. There exists a $u_0 > 0$ with the property: If $y \in (-d, d)$, $S_{d,u_0} \subset \Omega$ and $B_{d,u_0} \cup B_{-d,u_0} \subset \partial\Omega$, then*

$$|\omega(iy, (\partial\Omega)^+, \Omega) - \omega(iy, (\partial S_d)^+, S_d)| < \epsilon. \tag{14}$$

In the original proof Ω is fixed. However, a close inspection of the proof shows that u_0 depends only on d, not on the set Ω and that (14) holds for all $u > u_0$. We will use a variation of Lemma 1.

For $w \in \mathbb{C}$, $d_1, d_2, u > 0$, we consider the rectangles

$$A(w, d_1, d_2, u) = \{x + iy : |x - \operatorname{Re} w| < u/2,\ \operatorname{Im} w - d_2 < y < \operatorname{Im} w + d_1\}.$$

Let also, for $A = A(w, d_1, d_2, u)$,

$$\partial_h A = \{x + iy : |x - \operatorname{Re} w| < u/2,\ y = \operatorname{Im} w - d_2 \text{ or } y = \operatorname{Im} w + d_1\},$$

be the horizontal border of A. Finally for $z \in \mathbb{C}$, let

$$\partial_z^+ \Omega = \partial \Omega \cap \{\zeta : \operatorname{Im} \zeta > \operatorname{Im} z\} \tag{15}$$

be the part of the border of Ω that lies above z. Note that when $z \in \mathbb{R}$ we have $\partial_z^+ \Omega = (\partial \Omega)^+$. Note also that if the distances of iy from the upper and lower parts of a strip are respectively d_1 and d_2, by applying standard conformal maps, one can see that

$$\omega(iy, (\partial S_d)^+, S_d) = \frac{d_2}{d_1 + d_2}. \tag{16}$$

By conformal invariance of the harmonic measure, Lemma 1 can be restated as follows.

Lemma 2. *Let $d_1, d_2 > 0$. Then for every $\epsilon > 0$, there exists a $u_0 > 0$, such that for every $u > u_0$ and for all domains Ω, convex in the positive direction, the following property holds: If $A = A(w, d_1, d_2, u) \subset \Omega$ and $\partial_h A \subset \partial \Omega$, then*

$$\left| \omega(w, \partial_w^+ \Omega, \Omega) - \frac{d_2}{d_1 + d_2} \right| < \epsilon. \tag{17}$$

We will be working with domains convex in the positive direction but we point out that by a small modification of the proof found in [4], we can drop this requirement.

Let $z \in \mathbb{D}$. We will prove that the slope of the trajectory γ_z of a semigroup of holomorphic functions (ϕ_t) is determined by certain harmonic measures. Consider the function

$$\omega_z(t) = \omega(h(z) + t, \partial_{h(z)}^+ \Omega, \Omega),\ t \in (0, +\infty). \tag{18}$$

Betsakos [4] constructed a semigroup such that for every $z \in \mathbb{D}$, $\texttt{Slope}^+(\gamma_z) = [-\pi/2, \pi/2]$, by considering the behavior of $\omega_0(t)$ as $t \to +\infty$. We will prove an explicit relation between the behavior of $\omega_z(t)$ and the slopes of γ_z. We will then use it to construct a semigroup such that $\texttt{Slope}^+(\gamma_z) = [\theta_1, \theta_2]$ with $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$. The same principles will be extended to an analogous result for the $\texttt{Slope}^-(\gamma_z)$.

Theorem 3. *Let (ϕ_t) be a semigroup of holomorphic functions in $\mathbb{D}$. Denote by h the corresponding Koenigs function and by $\Omega = h(\mathbb{D})$ the associated planar domain. For $z \in \mathbb{D}$, with $\partial_{h(z)}^+ \Omega \neq \emptyset$ and $\partial_{h(z)}^+ \Omega \neq \partial \Omega$, let $a_1 = \limsup_{t \to \infty} \omega_z(t)$ and $a_2 = \liminf_{t \to \infty} \omega_z(t)$. Then*

$$\texttt{Slope}^+(\gamma_z) = [\pi(1/2 - a_1),\ \pi(1/2 - a_2)]\,. \tag{19}$$

If, in addition, for that z, the trajectory γ_z is defined for all $t \in (-\infty, 0]$ and we have $b_1 = \limsup_{t \to -\infty} \omega_z(t)$ and $b_2 = \liminf_{t \to -\infty} \omega_z(t)$, then

$$\texttt{Slope}^-(\gamma_z) = [\pi(1/2 - b_1),\ \pi(1/2 - b_2)]\,. \tag{20}$$

Using the above theorem we can argue about the slopes of the trajectories of (ϕ_t) by focusing on the image $h(\mathbb{D})$ and looking at the behavior of the harmonic measure on the points of the half-line $\{h(z) + t : t > 0\}$, or on $\{h(z) - t : t > 0\}$ for the backward trajectories.

3. Proofs

Proof (Theorem 3). We assume that the Denjoy-Wolff point of (ϕ_t) is ξ and the α-limit of γ_z is χ. Let $\widehat{\chi\xi}$ be the arc on $\partial\mathbb{D}$ between χ and ξ, corresponding through $h(z)$ to $\partial^+_{h(z)}\Omega$. Note that $\partial^+_{h(z)}\Omega \neq \emptyset$ and $\partial^+_{h(z)}\Omega \neq \partial\Omega$ imply $\chi \neq \xi$. Also since h is conformal we have that $\widehat{\chi\xi}$ is the arc that runs clockwise from χ to ξ. We know that the level set

$$L_k = \{\zeta \in \mathbb{D} : \omega(\zeta, \widehat{\chi\xi}, \mathbb{D}) = k\},\ 0 < k < 1, \tag{21}$$

is a circular arc with endpoints χ and ξ that meets the unit circle with angle $k\pi$. Let $\widehat{L}_k = \{\zeta \in \mathbb{D} : \omega(\zeta, \widehat{\chi\xi}, \mathbb{D}) > k\}$ and Γ_k be the half-line emanating from ξ that is tangent to L_k at ξ. If ζ lies on Γ_k then $\arg(1 - \overline{\xi}\zeta) = \pi/2 - \pi k = \pi(1/2 - k)$.

By conformal invariance of the harmonic measure (7),

$$\omega_z(t) = \omega(h(z) + t, \partial^+_{h(z)}\Omega, \Omega) = \omega(\phi_t(z), \widehat{\chi\xi}, \mathbb{D}). \tag{22}$$

Let $a_1 = \limsup_{t\to\infty} \omega_z(t)$ and $\theta_1 = \pi(1/2 - a_1)$ the corresponding angle.

We will prove that $\theta_1 = \min\{\texttt{Slope}^+(\gamma_z)\}$.

Claim 1. If $\theta \in \texttt{Slope}^+(\gamma_z)$ then $\theta_1 \leq \theta$.

If $a_1 = 1$ then $\theta_1 = -\pi/2$ and we are done. If not, since $a_1 = \limsup_{t\to\infty} \omega_z(t)$, from (22) we must also have

$$\limsup_{t\to\infty} \omega(\phi_t(z), \widehat{\chi\xi}, \mathbb{D}) = a_1. \tag{23}$$

Assume that $\theta \in \texttt{Slope}(\gamma_z)$ with $\theta_1 > \theta = \pi(1/2 - a)$. So there is an $\epsilon > 0$ such that $a_1 < a_1 + \epsilon/2 < a_1 + \epsilon < a$. Then there is a sequence $t_n \to \infty$ such that all but finite of the points $\phi_{t_n}(z)$ lie above $\Gamma_{a_1+\epsilon}$ for some $\epsilon > 0$. This means that $\phi_{t_n}(z) \in \widehat{L}_{a_1+\epsilon/2}$ for almost all n. This implies that $\lim_{t_n\to\infty} \omega(\phi_{t_n}(z), \widehat{\chi\xi}, \mathbb{D}) \geq a_1 + \epsilon/2$, a contradiction. So $\theta_1 \leq \theta$.

Claim 2. $\theta_1 \in \texttt{Slope}^+(\gamma_z)$.

Since there exists t_n with $\omega(\phi_{t_n}(z), \widehat{\chi\xi}, \mathbb{D}) \to a_1$ we have that $\arg(1 - \overline{\xi}\phi_{t_n}(z)) \to \theta_1$ and so $\theta_1 \in \texttt{Slope}^+(\gamma_z)$.

We have shown that $\theta_1 = \min\{\texttt{Slope}^+(\gamma_z)\}$. Using the same arguments we can show that if $a_2 = \liminf_{t\to\infty} \omega_z(t)$ and $\theta_2 = \pi(1/2 - a_2)$ we have $\theta_2 = \max\{\texttt{Slope}^+(\gamma_z)\}$. This means that $\texttt{Slope}^+(\gamma_z) = [\pi(1/2 - a_1),\ \pi(1/2 - a_2)]$.

In the case when the α-limit of γ_z is a super-repulsive point, replacing ∞ with $-\infty$ and ξ with χ, using the same arguments, we obtain relation (20) for the $\texttt{Slope}^-(\gamma_z)$.

Remark 1. The only property of the set $\partial^+_{h(z)}\Omega$ that we use is that it corresponds, through h^{-1}, to an arc $\widehat{\chi\xi}$ on $\partial\mathbb{D}$ with ξ being the Denjoy-Wolff point, or χ being the α-limit of γ_z, and $\chi \neq \xi$. This means that even when $\partial^+_{h(z)}\Omega = \emptyset$ or $\partial^+_{h(z)}\Omega = \partial\Omega$ we can use the same approach by choosing a suitable subset of $\partial\Omega$.

Proof (Theorem 1). We will only prove the result for $|\theta_1|, |\theta_2| < \pi/2$ for simplicity. Small variations of the proof can also account for the cases of $\theta_1 = -\pi/2$ or $\theta_2 = \pi/2$. We will essentially present these variations in the proof of Theorem 2. We will modify the construction found in [4] and construct a set Ω such that for the associated semigroup we have $\texttt{Slope}^+(\gamma_0) = [\theta_1, \theta_2]$. Let $E[\zeta] = \{\zeta + t : t \leq 0\}$ be the half-line, parallel to the real axis, starting from ζ and extending to the left. Let $a_1 = \frac{1}{2} - \frac{\theta_1}{\pi}$ and $a_2 = \frac{1}{2} - \frac{\theta_2}{\pi}$, so that $0 < a_2 < a_1 < 1$. Let r_n, ρ_n be sequences such that

$$r_n = \frac{1-a_1}{a_1}\rho_n \tag{24}$$

and

$$r_n = \frac{1-a_2}{a_2}\rho_{n-1}, \ n \geq 2. \tag{25}$$

Since $a_1 > a_2$, both r_n and ρ_n are increasing. Note that these depend only on the choice of a_1, a_2 and r_1. For example $a_1 = \dfrac{3}{4}$, $a_2 = \dfrac{1}{3}$ and $r_1 = 6$ gives

$$r_n = 6^n \text{ and } \rho_n = 3 \cdot 6^n.$$

It is easy to see that definitions (24) and (25) indeed give

$$\frac{\rho_n}{\rho_n + r_n} = a_1 \text{ and } \frac{\rho_{n-1}}{\rho_{n-1} + r_n} = a_2. \tag{26}$$

Note that for $w = 0$ we have $\partial^+_w\Omega = (\partial\Omega)^+$ and choose an increasing sequence u'_n from Lemma 2, such that the following hold:

When $n = 2k - 1$, for all Ω with $A = A(w, r_k, \rho_k, u'_n) \subset \Omega$ and $\partial A_h \subset \partial\Omega$,

$$|\omega(w, (\partial\Omega)^+, \Omega) - \frac{\rho_k}{\rho_k + r_k}| < \frac{1}{n} \tag{27}$$

and for all Ω with $A = A(x, r_{k+1}, \rho_k, u'_n) \subset \Omega$ and $\partial A_h \subset \partial\Omega$,

$$|\omega(w, (\partial\Omega)^+, \Omega) - \frac{\rho_k}{\rho_k + r_{k+1}}| < \frac{1}{n}. \tag{28}$$

When $n = 2k$, for all Ω with $A = A(x, r_{k+1}, \rho_k, u'_n) \subset \Omega$ and $\partial A_h \subset \partial\Omega$,

$$|\omega(w, (\partial\Omega)^+, \Omega) - \frac{\rho_k}{\rho_k + r_{k+1}}| < \frac{1}{n} \tag{29}$$

and for all Ω with $A = A(x, r_{k+1}, \rho_{k+1}, u'_n) \subset \Omega$ and $\partial A_h \subset \partial\Omega$,

$$|\omega(w, (\partial\Omega)^+, \Omega) - \frac{\rho_{k+1}}{\rho_{k+1} + r_{k+1}}| < \frac{1}{n}. \tag{30}$$

Consider the partial sums $u_n = \sum_{j=1}^{n} u'_j$ and set

$$\Omega = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} (E[u_{2k-1} + ir_k] \cup E[u_{2k} - i\rho_k]). \tag{31}$$

The way Ω was constructed we have that Ω is convex in the positive direction. We also have that, for $n = 2k-1$, for the rectangles $A = A(x_n, r_k, \rho_k, u'_n)$ we have $A \subset \Omega$ and $\partial A_h \subset \partial\Omega$, where $x_n = (u_n + u_{n-1})/2$. Obviously $x_n \to \infty$. For $n = 2k$ the same holds for $A = A(x_n, r_{k+1}, \rho_k, u'_n)$.

So for $n = 2k-1$, from relations (26) and (27), we have,

$$|\omega(x_n, (\partial\Omega)^+, \Omega) - a_1| < \frac{1}{n} \tag{32}$$

and for $n = 2k$, from relations (26) and (29),

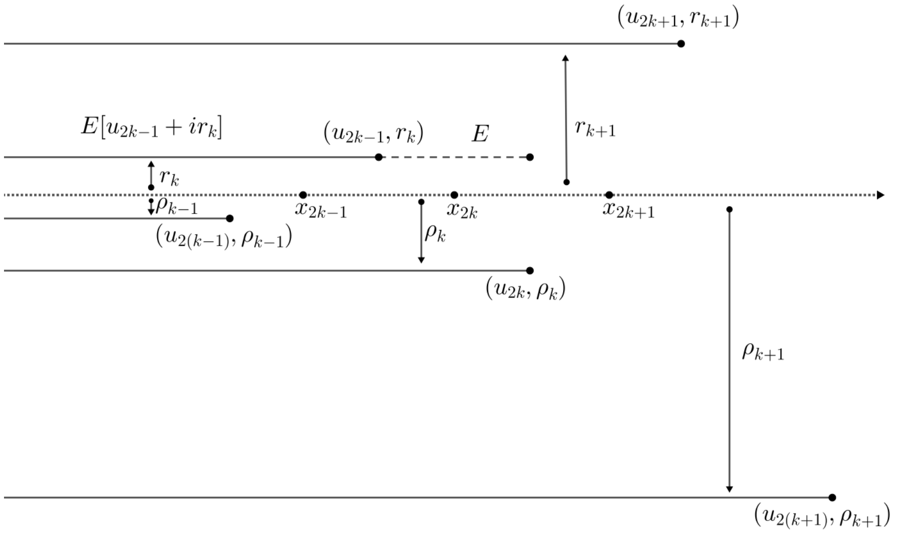
$$|\omega(x_n, (\partial\Omega)^+, \Omega) - a_2| < \frac{1}{n}. \tag{33}$$

So we have found two sequences $x_{2k-1} \in \mathbb{R}$ and $x_{2k} \in \mathbb{R}$ with respective limits a_2 and a_1. That means

$$[a_2, a_1] \subset [\liminf_{t\to\infty} \omega_0(t), \limsup_{t\to\infty} \omega_0(t)]. \tag{34}$$

We proceed to show the opposite inclusion. Consider a pair x_{2k-1}, x_{2k} on the real line. Note that the rectangles $A(x_{2k-1}, r_k, \rho_k, u'_{2k-1})$ and $A(x_{2k}, r_{k+1}, \rho_k, u'_{2k})$ are both contained in Ω.

Figure 1: A part of the set Ω



Consider the set $\Omega_1 = \Omega \setminus E$, where $E = \{x + iy : y = r_k,\ u_{2k-1} < x \le u_{2k}\}$. In Figure 2, E is the dotted segment. Obviously $\Omega_1 \subset \Omega$ and $(\partial\Omega)^- = (\partial\Omega_1)^-$. Also for all $x \in [x_{2k-1}, x_{2k}]$, since u'_n is increasing, we have that

$$A = A(x, r_k, \rho_k, u'_{2k-1}) \subset \Omega_1 \text{ and } \partial A_h \subset \partial\Omega_1. \tag{35}$$

Using the domain monotonicity of the harmonic measure and relation (27) we get

$$\begin{aligned}\omega(x,(\partial\Omega)^+,\Omega) = 1-\omega(x,(\partial\Omega)^-,\Omega) &\leq 1-\omega(x,(\partial\Omega_1)^-,\Omega_1)\\ &= \omega(x,(\partial\Omega_1)^+,\Omega_1) < a_1+\frac{1}{n}.\end{aligned}$$

Similarly consider $\Omega_2 = \Omega \cup E[u_{2k-1}+ir_k]$. Again for all $x \in [x_{2k-1}, x_{2k}]$, we have $A = A(x, r_{k+1}, \rho_k, u'_{2k-1}) \subset \Omega_2$ and $\partial A_h \subset \partial\Omega_2$. Since $\Omega \subset \Omega_2$, considering (28),

$$\omega(x,(\partial\Omega)^+,\Omega) > \omega(x,(\partial\Omega_2)^+,\Omega_2) > a_2-\frac{1}{n}.$$

We can likewise treat the case where $x \in [x_{2k}, x_{2k+1}]$. These inequalities show that if there exists a sequence $t_k \to \infty$ with $\lim_{k\to\infty}\omega_0(t_k) = a$ then $a_2 \leq a \leq a_1$.

We have shown that $a_1 = \limsup_{t\to\infty}\omega_0(t)$ and $a_2 = \liminf_{t\to\infty}\omega_0(t)$. Considering the semigroup (ϕ_t) that corresponds to the set Ω, the desired result follows from Theorem 3.

PROOF (THEOREM 2). As in the above proof let $b_1 = \frac{1}{2}-\frac{\theta_1}{\pi}$, $b_2 = \frac{1}{2}-\frac{\theta_2}{\pi}$ and r_n, ρ_n be sequences such that

$$r_n = \frac{1-b_2}{b_2}\rho_n,$$

and

$$r_n = \frac{1-b_1}{b_1}\rho_{n-1},\ \ n \geq 2.$$

Since $b_1 > b_2$ we have that both r_n and ρ_n are decreasing sequences. Note that these depend only on the choice of b_1, b_2 and r_1. Similar to the above proof, if for example $b_1 = \dfrac{3}{4}$, $b_2 = \dfrac{1}{3}$ and $r_1 = \dfrac{1}{3}$, we get

$$r_n = \frac{1}{3}\cdot 6^{-(n-1)} \text{ and } \rho_n = 6^{-n}.$$

We define sequences u_n, u'_n in the exact same way as in the proof of Theorem 1. This means that we can use relations (27 - 30). Now Ω can be defined as

$$\Omega = \mathbb{C}\setminus\bigcup_{n=1}^{\infty}(E[-u_{2k-1}+ir_k]\cup E[-u_{2k}+i\rho_k]).$$

Obviously Ω is convex in the positive direction and γ_0 is defined for $t \in (-\infty,+\infty)$. Similarly with before we take $x_n = -(u_n+u_{n-1})/2$. We have that x_n goes to $-\infty$ and for the subsequences x_{2k-1} and x_{2k} we get

$$\lim_{k\to\infty}\omega(x_{2k-1},(\partial\Omega)^+,\Omega) = b_1 \text{ and}$$
$$\lim_{k\to\infty}\omega(x_{2k},(\partial\Omega)^+,\Omega) = b_2.$$

We can show the opposite inclusion with the same arguments as in the proof of Theorem 1. Again from Theorem 3 we get $\texttt{Slope}^-(\gamma_0) = [\theta_1,\theta_2]$.

We will now consider the case when $b_2 = 0$. We modify our sequences so that

$$r_n = (n+m)\rho_n,$$

and

$$r_n = \frac{1-b_1}{b_1},\ n \geq 2,$$

where m is taken big enough, so that for all n we have $n + m > \frac{1-b_2}{b_2}$. We again have two decreasing sequences. The proof works out in the same way except that now, for $n = 2k-1$, relation (27) becomes

$$\omega(x_n, (\partial\Omega)^+, \Omega) < \frac{1}{n+m+1} + \frac{1}{n} < \frac{2}{n} \tag{36}$$

for all n. Obviously $\omega(x_{2k-1}, (\partial\Omega)^+, \Omega) \to 0$ as $k \to \infty$ and as before we have $\omega(x_{2k}, (\partial\Omega)^+, \Omega) \to b_1$. Similarly in the case when $b_1 = 1$ we take

$$r_n = \frac{1-b_2}{b_2}\rho_n,$$

and

$$r_n = \frac{1}{n+m},\ n \geq 2,$$

where m is taken big enough, so that for all n we have $\dfrac{1}{n+m} < \dfrac{1-b_2}{b_2}$. As before, note that, for $n = 2k$, relation (29) becomes

$$\begin{aligned}\omega(x_n, \partial\Omega^+, \Omega) &> \frac{n+m}{n+m+1} - \frac{1}{n}\\ &> \frac{n+m}{n+m+1} - \frac{2}{n+m+1} = 1 - \frac{3}{n+m+1},\end{aligned}$$

for all $n > m+1$. Obviously $\omega(x_{2k}, (\partial\Omega)^+, \Omega) \to 1$ as $k \to \infty$, while $\omega(x_{2k-1}, (\partial\Omega)^+, \Omega) \to b_2$.

Combining the above we can also construct an example with $\texttt{Slope}^-(\gamma_z) = [-\pi/2, \pi/2]$. Note that in this case we can simply use

$$r_n = n\rho_n,$$

and

$$r_n = \frac{1}{n}\rho_{n-1},\ n \geq 2,$$

which coincides with what was used in [4].

Acknowledgements

I would like to thank professor D. Betsakos, my thesis advisor, for his help.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declarations of interest: none.

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