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EQUALITY CASE FOR AN ELLIPTIC AREA CONDENSER INEQUALITY AND A RELATED SCHWARZ TYPE LEMMA

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Abstract We give an equality condition for a symmetrization inequality for condensers proved by F.W. Gehring regarding elliptic areas. We then use this to obtain a monotonicity result involving the elliptic area of the image of a holomorphic function f.

Keywords: condenser, capacity, conformal map, elliptic area, symmetrization

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1. Introduction

A condenser is a pair (D, K) where D is an open set in $\mathbb{C}$ and K is a non-empty compact subset of D. The open set $D \setminus K$ is called the field of the condenser and the sets $\mathbb{C} \setminus D$ and K are called the plates of the condenser. The capacity of a condenser is defined as

$$\operatorname{cap}(D, K) = \inf_u \int_{D\setminus K} |\nabla u|^2 d\sigma,$$

where the infinum is taken over all Lipschitz functions in $D \setminus K$ with boundary limits 1 on ∂D and 0 on ∂K and σ is the standard two-dimensional Lebesgue measure, see [6] for details. A condenser is called admissible if there exists a continuous real function $\omega(z)$ in $\mathbb{C}_\infty$ that is equal to 0 on ∂K, to 1 on ∂D and is harmonic in $D \setminus K$. It is known that if $r\mathbb{D} = \{z \in \mathbb{C} : |z| \leq r\}$, then

$$\operatorname{cap}(s\mathbb{D}, r\mathbb{D}) = 2\pi \left(\log \frac{s}{r}\right)^{-1}. \tag{1.1}$$

Let (D, K) be a condenser such that for every connected component D_i of D the logarithmic capacity of $D_i \cap K$ is positive. Let f be a meromorphic map which is non-constant on every connected component of D . Then $(f(D), f(K))$ is also a condenser and

$$\operatorname{cap}(D, K) \geq \operatorname{cap}(f(D), f(K)),$$

with equality if and only if f is a conformal map [12].

Generally the further apart ∂D and ∂K are, the smaller the capacity. That implies that the more symmetric the condenser is, the smaller its capacity. For more results regarding condensers and symmetrized sets we refer to [6] and [7].

Pólya and Szegö [11, p. 191], have shown that if K^* and D^* are concentric disks having the same Euclidean area with K and D then

$$\operatorname{cap}(D,K) \geq \operatorname{cap}(D^*,K^*).$$

F.W. Gehring gives a proof of the above in [7], also showing that the same is true if we consider the hyperbolic or the elliptic areas instead of the Euclidean. The equality conditions were studied in [3] for the Euclidean and in [2] for the hyperbolic case. We will prove an analogous condition for the elliptic case.

In $\mathbb{C}_\infty$ we define the elliptic area of a set Ω using the metric with density $\frac{1}{1+|z|^2}$. We use the notation $A_e(\Omega)$. The distance function for the above metric is $\mathrm{d}_e[z,w] = \arctan\left|\frac{z-w}{1+z\bar{w}}\right|$.

One can identify $\mathbb{C}_\infty$ with the unit sphere in $\mathbb{R}^3$ by means of a suitable projection, namely ϕ with $\phi(\infty) = (0,0,1)$ and

$$\phi(z) = \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{2\operatorname{Im} z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right), \ z \in \mathbb{C}. \tag{1.2}$$

We then have that the quantity $\chi(z,w) = 2\sin(\mathrm{d}_e[z,w])$, called chordal distance, equals the distance of $\phi(z)$ and $\phi(w)$ in $\mathbb{R}^3$ [9].

Note that disks in $\mathbb{C}$ map to spherical caps on the sphere and spherical caps not containing $\phi(\infty) = (0,0,1)$ are images of disks in $\mathbb{C}$. Also straight lines map to circles passing through $(0,0,1)$ on the sphere. Since $\phi(0) = (0,0,-1)$ the real line maps to the great circle $\{(x,0,\sqrt{1-x^2}) : x \in \mathbb{R}\}$. The maximum distance of two points is $\frac{\pi}{2}$. When $\mathrm{d}_e[z,w] = \frac{\pi}{2}$ we call z and w antipodal. It is easy to see that the antipodal point of z is $-1/\bar{z}$ and that the images through ϕ of two antipodal numbers are antipodal points of the sphere.

Elliptic area is invariant under conformal automorphisms of the form [9, p.178]

$$\phi(z) = \frac{bz - \bar{c}}{cz + \bar{b}}, \ |b|^2 + |c|^2 > 0. \tag{1.3}$$

By setting $b = 1$ and $c = \bar{a}$ in (1.3) we define

$$\phi_a(z) = \frac{z-a}{1+\bar{a}z}.$$

Obviously ϕ_a is an elliptic isometry sending a to 0. Note that the inverse of ϕ_a is $(\phi_a^{-1})(z) = \frac{z+a}{1-\bar{a}z} = \phi_{-a}(z)$ and we have

$$\mathrm{d}_e[z,a] = \arctan|\phi_a(z)|. \tag{1.4}$$

We will use the notation Ω° to denote the open disk centered at the origin having the same elliptic area as the set Ω, in the case when Ω is open. When Ω is closed, Ω° will denote the closed disk of the same elliptic area. So $A_e(\Omega) = A_e(\Omega^\circ)$.

F.W. Gehring has shown that [7]

Theorem 1.1 (Gehring)**.** *Let (D,K) be a condenser and D°, K° the disks centered at the origin with $A_e(D) = A_e(D^\circ)$ and $A_e(K) = A_e(K^\circ)$. We have*

$$\mathrm{cap}(D^\circ, K^\circ) \leq \mathrm{cap}(D,K). \tag{1.5}$$

We will study the equality case for the above inequality, proving that

Theorem 1.2. *Let (D,K) be a condenser and D°, K° as defined previously. Suppose, in addition, that (D,K) is an admissible condenser. Then equality in (1.5) holds if and only if there exists an elliptic isometry ϕ_c such that $D^\circ = \phi_c(D)$ and $K^\circ = \phi_c(K)$.*

The second result is a monotonicity theorem in the spirit of [4]. One can see the classical Schwarz lemma as a monotonicity result stating that if the function f is non-constant and holomorphic on the unit disk and $\mathrm{Rad}(f(r\mathbb{D})) := \sup_{z<r} |f(z) - f(0)|$ is the “radius” of $f(r\mathbb{D})$, then the function

$$\frac{\mathrm{Rad}(f(r\mathbb{D}))}{r}, \ 0 < r < 1,$$

is an increasing function of r. In that spirit, Burckel et al. have proven analogous results where the radius is replaced by other geometric quantities such as the diameter or the capacity. An area version can be found in [1] and Betsakos has studied the cases of hyperbolic-area-radius and hyperbolic capacity in [2]. Further results were obtained more recently by other authors, see for example [5] and [8].

We will prove an analogous result for the elliptic-area-radius of a set Ω. It is defined precisely as the Euclidean radius of the disk Ω°. We will be using the symbol $R_e(\Omega)$. This is in accordance with Pólya and Szegö in [11, p. 4] and also Betsakos [2]. The elliptic area of a disk of radius r centered at the origin is equal with

$$A_e(r\mathbb{D}) = \frac{\pi r^2}{1+r^2}. \tag{1.6}$$

From the above we have that

$$R_e(\Omega) = \left(\frac{A_e(\Omega)}{\pi - A_e(\Omega)}\right)^{\frac{1}{2}}. \tag{1.7}$$

We also note that $A_e(\mathbb{C}) = \pi$.

We use ϕ_a to define elliptic disks centered at a as

$$\Delta(a,r) = \{z \in \mathbb{C} : |\phi_a(z)| < r\}.$$

In the case $a = 0$, we have that $\Delta(0,r) = r\mathbb{D}$ and so relation (1.1) becomes

$$\mathrm{cap}\left(\Delta(0,s), \Delta(0,r)\right) = 2\pi \left(\log\frac{s}{r}\right)^{-1}. \tag{1.8}$$

We can see that, with the above notation, $\Omega^\circ = \Delta(0, R_e(\Omega))$. Also

$$\phi_a(\Delta(a,r)) = \Delta(0,r) \text{ and } \phi_{-a}(\Delta(0,r)) = \Delta(a,r). \tag{1.9}$$

Since ϕ_a is an isometry, we have $A_e(\Delta(a,r)) = A_e(\Delta(0,r))$ which means that $R_e\Delta(a,r) = r$.

Finally we define the elliptic distance $\mathrm{dist}_e(a, \partial\Omega)$ of $a \in \Omega$ from $\partial\Omega$ as the infimum of $\{\mathrm{d}_e[a,\zeta] : \zeta \in \partial\Omega\}$. Note that the elliptic radius of $\Delta(a,r)$ equals that of $\Delta(0,r)$, namely $\arctan r$. This implies that for $r < \tan(\mathrm{dist}_e(a,\partial\Omega))$ we have that, for $z \in \Delta(a,r)$, $\mathrm{d}_e[a,z] < \arctan r < \mathrm{dist}_e(a,\partial\Omega)$. This means that $\Delta(a,r) \subset \Omega$.

Theorem 1.3. *Let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function from a domain Ω to $\mathbb{C}$. Fix $a \in \Omega$ and let $\rho = \tan(\mathrm{dist}_e(a,\partial\Omega))$.*

(i) *The function*

$$\Pi(r) = \frac{R_e f(\Delta(a,r))}{r}, \ 0 < r < \rho$$

is increasing.

(ii) *The function $\Pi(r)$ is strictly increasing unless $f = \phi_{-b} \circ F \circ \phi_a$, where $F(z) = \lambda z$ and λ, b are constants in $\mathbb{C}$.*

(iii)

$$\lim_{r \to 0^+} \Pi(r) = \frac{1+|a|^2}{1+|f(a)|^2}|f'(a)|.$$

(iv)

$$|f'(a)| \leq \frac{1}{\rho} \cdot \frac{1+|f(a)|^2}{1+|a|^2} \left(\frac{A_e f(\Delta(a,\rho))}{\pi - A_e f(\Delta(a,\rho))} \right)^{\frac{1}{2}}.$$

The elliptic derivative of f is defined using the chordal distance as $\lim_{|w-z|\to 0} \frac{\chi(z,w)}{|x-z|}$. It is equal with

$$f^{\#}(z) = \frac{|f'(a)|}{1+|f(a)|^2}.$$

Inequality (iv) can be restated as

$$|f^{\#}(a)| \leq \frac{1}{(1+|a|^2)\rho} \left(\frac{A_e f(\Delta(a,\rho))}{\pi - A_e f(\Delta(a,\rho))} \right)^{\frac{1}{2}}. \tag{1.10}$$

Bounds for this derivative have been given by S. Yamashita. One of these [13] states that

$$f^{\#}(a) \leq \left(\frac{A_e f(\Omega)}{\pi - A_e f(\Omega)} \right)^{\frac{1}{2}} P_\Omega(a),$$

where $P_\Omega(a)$ is the hyperbolic density of Ω at point a. Substituting $\Delta(a,\rho)$ for Ω and calculating $P_{\Delta(a,\rho)}(a)$, we obtain inequality (1.10).

2. Lemmas

We will need the notion of circular symmetrization for the proofs. A general reference for symmetrizations is [6]. The circular symmetrization of an open set Ω with respect to the positive semi-axis is defined as

$$\mathrm{Cr}\,\Omega = \{re^{i\theta} : \Omega \cap \gamma(r) \neq 0, 2|\theta|r < m(\Omega \cap \gamma(r)\} \cup \{-r : \gamma(r) \subset \Omega\},$$

where m is the arc-length measure and $\gamma(r) = \{z \in \mathbb{C} : |z| = r\}$. Similarly for a closed set E we have

$$\mathrm{Cr}\,E = \{re^{i\theta} : E \cap \gamma(r) \neq 0, 2|\theta|r \leq m(E \cap \gamma(r)\} \cup \{-r : \gamma(r) \subset E\}.$$

By means of a suitable motion ϕ, we define circular symmetrization with respect to an arbitrary ray L. Circular symmetrization doesn't change Euclidean area. When the ray L eminates from the origin, due to its radial nature, it is one of the few symmetrizations that also preserves hyperbolic and elliptic area. It is known that circular symmetrization reduces condenser capacity [6, p.94]. We proceed with some necessary lemmas.

In the case D is not connected, a condenser (D,K) can be seen as the sum of two condensers. If D_1 is a connected component of D and $K_1 = D_1 \cap K$, then

$$\mathrm{cap}(D,K) = \mathrm{cap}(D_1,K_1) + \mathrm{cap}(D \setminus D_1, K \setminus K_1).$$

We will need the following lemma to prove that in this case inequality (1.5) is strict.

Lemma 2.1. *Let* $(D,K) = (\mathit{\Delta}(0,s), \overline{\mathit{\Delta}(0,r)})$ *be a condenser. Also let* $(D_i,K_i) = (\mathit{\Delta}(0,s_i), \overline{\mathit{\Delta}(0,r_i)})$, $r_i < s_i$, $i = 1,2$ *be two condensers such that* $A_e(D) = A_e(D_1) + A_e(D_2)$ *and* $A_e(K) = A_e(K_1) + A_e(K_2)$. *Then*

$$\mathrm{cap}(D,K) < \mathrm{cap}(D_1,K_1) + \mathrm{cap}(D_2,K_2).$$

Proof.
We have $A_e(\mathit{\Delta}(0,s)) = A_e(\mathit{\Delta}(0,s_1)) + A_e(\mathit{\Delta}(0,s_2))$ and so

$$\frac{\pi s_1^2}{1+s_1^2} + \frac{\pi s_2^2}{1+s_2^2} = \frac{\pi s^2}{1+s^2}.$$

Solving for s we obtain

$$s^2 = \frac{s_1^2 + 2s_1^2 s_2^2 + s_2^2}{1 - s_1^2 s_2^2}.$$

Similarly

$$r^2 = \frac{r_1^2 + 2r_1^2 r_2^2 + r_2^2}{1 - r_1^2 r_2^2}.$$

Note that the formulas for s and r imply that $1 - s_1^2 s_2^2 > 0$ and $1 - r_1^2 r_2^2 > 0$. Setting $k = s_1/r_1$, we have $s_1 = kr_1$ and $k > 1$. We can assume without loss of generality that $s_1/r_1 \leq s_2/r_2$, which gives $s_2/k \geq r_2$. Since $r_i < s_i$, $k > 1$ and $s_2/k \geq r_2$ we have

$$1 - s_1^2 s_2^2 < 1 - r_1^2 r_2^2$$

and

$$r_1^2 + 2r_1^2 s_2^2 + s_2^2/k^2 > r_1^2 + 2r_1^2 r_2^2 + r_2^2.$$

This gives

$$\frac{(r_1^2 + 2r_1^2 s_2^2 + s_2^2/k^2)(1 - r_1^2 r_2^2)}{(1 - s_1^2 s_2^2)(r_1^2 + 2r_1^2 r_2^2 + r_2^2)} > 1. \tag{2.1}$$

Because of (1.8) and (2.1), we have

$$\begin{aligned}
\frac{1}{2\pi}\operatorname{cap}(\mathit{\Delta}(0,s), \mathit{\Delta}(0,r)) &= (\log \frac{s}{r})^{-1} = (\frac{1}{2}\log\frac{s^2}{r^2})^{-1} \\
&= 2\left(\log \frac{(s_1^2 + 2s_1^2 s_2^2 + s_2^2)(1 - r_1^2 r_2^2)}{(1 - s_1^2 s_2^2)(r_1^2 + 2r_1^2 r_2^2 + r_2^2)}\right)^{-1} \\
&= 2\left(\log \frac{(k^2 r_1^2 + 2k^2 r_1^2 s_2^2 + s_2^2)(1 - r_1^2 r_2^2)}{(1 - s_1^2 s_2^2)(r_1^2 + 2r_1^2 r_2^2 + r_2^2)}\right)^{-1} \\
&= 2\left(\log \left(k^2 \frac{(r_1^2 + 2r_1^2 s_2^2 + s_2^2/k^2)(1 - r_1^2 r_2^2)}{(1 - s_1^2 s_2^2)(r_1^2 + 2r_1^2 r_2^2 + r_2^2)}\right)\right)^{-1} \\
&< 2(\log k^2)^{-1} = (\log k)^{-1} = \frac{1}{2\pi}\operatorname{cap}(D_1, K_1) \\
&< \frac{1}{2\pi}(\operatorname{cap}(D_1, K_1) + \operatorname{cap}(D_2, K_2)).
\end{aligned}$$

□

We will also need to prove that inequality in (1.5) is strict in the case when both D and K are annuli centered at the origin.

Lemma 2.2. *Let* $D = \mathit{\Delta}(0, s_1) \setminus \overline{\mathit{\Delta}(0, s_2)}$ *and* $K = \overline{\mathit{\Delta}(0, r_1)} \setminus \mathit{\Delta}(0, r_2)$, *with* $s_2 < r_2 < r_1 < s_1$. *Then*

$$\operatorname{cap}(D^\circ, K^\circ) < \operatorname{cap}(D, K).$$

Proof. Since $s_2 < r_2 < r_1 < s_1$ we have that $K \subset D$ and (D, K) is indeed a condenser. Consider the condensers $(D_1, K_1) = (\mathit{\Delta}(0, s_1), \overline{\mathit{\Delta}(0, r_1)})$ and $(D_2, K_2) = (\mathit{\Delta}(0, r_2), \overline{\mathit{\Delta}(0, s_2)})$ and note that

$$\operatorname{cap}(D, K) = \operatorname{cap}(D_1, K_1) + \operatorname{cap}(D_2, K_2).$$

Now let $s = R_e(D)$ and $r = R_e(K)$, so that $D^\circ = \mathit{\Delta}(0, s)$ and $K^\circ = \overline{\mathit{\Delta}(0, r)}$. We have $A_e(\mathit{\Delta}(0, s)) = A_e(\mathit{\Delta}(0, s_1)) - A_e(\mathit{\Delta}(0, s_2))$ and so

$$\frac{\pi s_1^2}{1 + s_1^2} - \frac{\pi s_2^2}{1 + s_2^2} = \frac{\pi s^2}{1 + s^2}.$$

Solving for s we obtain

$$s^2 = \frac{s_1^2 - s_2^2}{1 + 2s_2^2 + s_1^2 s_2^2}.$$

Similarly

$$r^2 = \frac{r_1^2 - r_2^2}{1 + 2r_2^2 + r_1^2 r_2^2}.$$

Setting $k = s_1/r_1$ and $l = r_2/s_2$, we have $s_1 = kr_1$ and $r_2 = ls_2$. First consider the case $k \le l$. Note that $r_1^2 - l^2 s_2^2 = r_1^2 - r_2^2 > 0$. Since $k, l > 1$ we have

$$r_1^2 - s_2^2/k^2 > r_1^2 - l^2 s_2^2$$

and

$$1 + 2l^2 s_2^2 + l^2 r_1^2 s_2^2 > 1 + 2s_2^2 + k^2 r_1^2 s_2^2.$$

This gives

$$\frac{(r_1^2 - s_2^2/k^2)(1 + 2l^2 s_2^2 + l^2 r_1^2 s_2^2)}{(1 + 2s_2^2 + k^2 r_1^2 s_2^2)(r_1^2 - l^2 s_2^2)} > 1. \tag{2.2}$$

Because of (1.8) and (2.2) we have

$$\begin{aligned}
\frac{1}{2\pi}\operatorname{cap}(D^\circ, K^\circ) = (\log\frac{s}{r})^{-1} = (\frac{1}{2}\log\frac{s^2}{r^2})^{-1} \\
&= 2\left(\log\frac{(s_1^2 - s_2^2)(1 + 2r_2^2 + r_1^2 r_2^2)}{(1 + 2s_2^2 + s_1^2 s_2^2)(r_1^2 - r_2^2)}\right)^{-1} \\
&= 2\left(\log\frac{(k^2 r_1^2 - s_2^2)(1 + 2l^2 s_2^2 + l^2 r_1^2 s_2^2)}{(1 + 2s_2^2 + k^2 r_1^2 s_2^2)(r_1^2 - l^2 s_2^2)}\right)^{-1} \\
&= 2\left(\log\left(k^2\frac{(r_1^2 - s_2^2/k^2)(1 + 2l^2 s_2^2 + l^2 r_1^2 s_2^2)}{(1 + 2s_2^2 + k^2 r_1^2 s_2^2)(r_1^2 - l^2 s_2^2)}\right)\right)^{-1} \\
&< 2(\log k^2)^{-1} = (\log k)^{-1} < (\log k)^{-1} + (\log l)^{-1} \\
&= \frac{1}{2\pi}\operatorname{cap}(D_1, K_1) + \frac{1}{2\pi}\operatorname{cap}(D_2, K_2) = \frac{1}{2\pi}\operatorname{cap}(D, K).
\end{aligned}$$

Similarly we treat the case $k > l$. We now have

$$\begin{aligned}
&(k^2/l^2)r_1^2 + 2k^2 r_1^2 s_2^2 + k^2 r_1^4 s_2^2 - s_2^2/l^2 - 2s_2^4 - r_1^2 s_2^4 \\
> \ & r_1^2 + 2r_1^2 s_2^2 + k^2 r_1^4 s_2^2 - l^2 s_2^2 - 2l^2 s_2^4 - l^2 k^2 r_1^2 s_2^4.
\end{aligned}$$

This gives

$$\frac{(k^2 r_1^2 - s_2^2)(1/l^2 + 2s_2^2 + r_1^2 s_2^2)}{(1 + 2s_2^2 + k^2 r_1^2 s_2^2)(r_1^2 - l^2 s_2^2)} > 1.$$

In this case we use l^2 as a common factor and we get

$$
\begin{aligned}
\frac{1}{2\pi}\operatorname{cap}(D^\circ, K^\circ) &= 2\left(\log\frac{(k^2r_1^2 - s_2^2)(1+2l^2s_2^2+l^2r_1^2s_2^2)}{(1+2s_2^2+k^2r_1^2s_2^2)(r_1^2-l^2s_2^2)}\right)^{-1} \\
&= 2\left(\log\left(l^2\frac{(k^2r_1^2 - s_2^2)(1/l^2+2s_2^2+r_1^2s_2^2)}{(1+2s_2^2+k^2r_1^2s_2^2)(r_1^2-l^2s_2^2)}\right)\right)^{-1} \\
&< (\log l)^{-1} < (\log k)^{-1} + (\log l)^{-1} \\
&= \frac{1}{2\pi}\operatorname{cap}(D_1, K_1) + \frac{1}{2\pi}\operatorname{cap}(D_2, K_2) = \frac{1}{2\pi}\operatorname{cap}(D, K).
\end{aligned}
$$

□

We know that when (D, K) is an admissible condenser with connected field

$$\operatorname{cap}(\operatorname{Cr} D, \operatorname{Cr} K) \le \operatorname{cap}(D, K)$$

with equality if and only if (D, K) is already circularly symmetric with respect to a ray emanating from the origin [6, p.94]. In order to use this fact we will need the following lemma regarding circular symmetrization.

Lemma 2.3. *Let* $\phi(z) = \phi_c(z) = \dfrac{z-c}{1+cz}$, $c \in \mathbb{R}$ *and* D *an either closed or open subset of* $\mathbb{C}$. *Suppose*

(i) *D is circularly symmetric with respect to the positive semi-axis.*

(ii) *$\phi(D)$ is up to rotation circularly symmetric with respect to the positive semi-axis,*

Then $\phi(D)$ is circularly symmetric with respect to either the positive or the negative semi-axis.

Proof. Assume D is open. We will use proof by contradiction. Let L_+ be the positive semi-axis and remember that $\gamma(r) = \{z \in \mathbb{C} : |z| = r\}$. First observe that ϕ^{-1} takes complex conjugates to complex conjugates. If for all $r > 0$, $\gamma(r) \cap \phi(D)$ is equal with either $\gamma(r)$ or $\emptyset$ then we are done. If not then there is at least one $r > 0$ such that $\gamma(r) \cap \phi(D)$ is a circular arc whose ends are not conjugate numbers. This arc has the form

$$\{re^{i\theta} : a - b < \theta < a + b\}, 0 < |a| < \pi, 0 < b \le \pi.$$

If $re^{-i(a+b)} \in \phi(D)$ and $b \neq \pi$ picking a sufficiently small ϵ, we can find

$$a_1 = re^{i(a+b+\epsilon)} \notin \phi(D) \text{ and } a_2 = re^{-i(a+b+\epsilon)} \in \phi(D).$$

Note that a_1 and a_2 are complex conjugates. Similarly, if $re^{-i(a+b)} \notin \phi(D)$ we have

$$a_2 = re^{i(a+b-\epsilon)} \in \phi(D) \text{ and } a_1 = re^{-i(a+b-\epsilon)} \notin \phi(D).$$

If $b = \pi$, then we have that $\gamma(r) \cap \phi(D) = \gamma(r) \setminus \{re^{i(a+b)}\}$. In this case set

$$a_1 = re^{i(a+b)} \notin \phi(D) \text{ and } a_2 = re^{-i(a+b)} \in \phi(D).$$

For all cases, set $b_1 = \phi^{-1}(a_1) \notin D$ and $b_2 = \phi^{-1}(a_2) \in D$. Now b_1 and b_2 are also conjugates and this contradicts the fact that D is circularly symmetric with respect to L_+. Similarly if D is closed. □

Lastly we prove the following for the sake of completeness.

Lemma 2.4. *If F maps conformally two concentric disks $\Delta(0, s)$ and $\Delta(0, r)$ onto two concentric disks $\Delta(0, s')$ and $\Delta(0, r')$ then $F(z) = \lambda z$, $\lambda \in \mathbb{C}$.*

Proof. Define

$$g(z) = \frac{F(sz)}{s'}.$$

Note that g maps $\mathbb{D}$ onto $\mathbb{D}$ conformally and it also maps the disk $\Delta(0, \frac{r}{s})$ onto the disk $\Delta(0, \frac{r'}{s'})$. It is therefore a Möbius Transformation and thus a hyperbolic isometry. Since hyperbolic isometries cannot change the hyperbolic diameter of a set and preserve the centers of hyperbolic circles, it follows that $\frac{r}{s} = \frac{r'}{s'}$ and also $g(0) = 0$. From the Schwarz lemma it follows that $g(z) = \lambda' z$ and so

$$F(z) = \lambda' \frac{s'}{s} z = \lambda z.$$

□

3. Proof of Theorem 1.2

Suppose that (D, K) is admissible and

$$\operatorname{cap}(D, K) = \operatorname{cap}(D^\circ, K^\circ).$$

Claim 3.1. *D is connected.*

Proof. Assume D is not connected and let D_1 be one of its components. Let $K_1 = D_1 \cap K$, $D_2 = D \setminus D_1$ and $K_2 = K \setminus K_1$. Note that K_1 cannot have zero logarithmic capacity. If that was the case, then $A_e(K_1) = 0$. Since $A_e(D) > A_e(D_2)$, for $s' = R_e(D_2)$ and $s = R_e(D)$ we have

$$\begin{aligned} \operatorname{cap}(D, K) &= \operatorname{cap}(D_2, K_2) \geq \operatorname{cap}(D_2^\circ, K_2^\circ) \\ &= 2\pi \left(\log \frac{s'}{r}\right)^{-1} > 2\pi \left(\log \frac{s}{r}\right)^{-1} = \operatorname{cap}(D^\circ, K^\circ). \end{aligned} \tag{3.1}$$

The same reasoning applies to K_2. So

$$\operatorname{cap}(D, K) = \operatorname{cap}(D_1, K_1) + \operatorname{cap}(D_2, K_2) \geq \operatorname{cap}(D_1^\circ, K_1^\circ) + \operatorname{cap}(D_2^\circ, K_2^\circ). \tag{3.2}$$

But since $A_e(D) = A_e(D_1) + A_e(D_2)$ and $A_e(K) = A_e(K_1) + A_e(K_2)$, by Lemma 2.1 and (3.2),

$$\operatorname{cap}(D, K) > \operatorname{cap}(D^\circ, K^\circ).$$

So D must be connected. □

Claim 3.2. *$D \setminus K$ is connected.*

Proof. We first show that all the connected components of K are simply connected. If K_1 is a component of K that is not simply connected, let K_c be the union of K_1 with all the bounded components of $\mathbb{C} \setminus K_1$. Then $K_h := K_c \setminus K_1$ is the union of the holes of K_1. Let $D_h := K_h \cap (\mathbb{C} \setminus D)$. Note that K_h is open and D_h is a compact subset of K_h. We first consider the case $D_h = \emptyset$. Then $\operatorname{cap}(D, K) = \operatorname{cap}(D, K \cup K_h)$ and using the same argument as in (3.1), we have $\operatorname{cap}(D, K) > \operatorname{cap}(D^\circ, K^\circ)$.

In the case $D_h \neq \emptyset$, we have that

$$\operatorname{cap}(D, K) = \operatorname{cap}(D \cup D_h, K \cup K_h) + \operatorname{cap}(K_h, D_h),$$

with $A_e(D) = A_e(D \cup D_h) - A_e(D_h)$ and $A_e(K) = A_e(K_c) - A_e(K_h)$. Using Lemma 2.2 and applying the same reasoning as in the proof of Claim 3.1, we get a contradiction and so all the connected components of K are simply connected, which means that $\mathbb{C} \setminus K$ is connected. By a consequence of a generalisation of Alexander's Lemma [10, Theorem 16.2] we have that since D and $\mathbb{C} \setminus K$ are connected and K is a bounded subset of D, then $D \setminus K$ is also connected. □

Completion of the proof of Theorem 1.2. Since (D, K) is an admissible condenser with connected field, circular symmetrization with respect to a ray emanating from the origin doesn't change elliptic area and decreases capacity [6]. We can assume that (D, K) is up to rotation circularly symmetric with respect to the positive semi-axis. If not, since (D, K) is an admissible condenser with connected field, we could apply a circular symmetrization and the resulting condenser would have equal elliptic areas with (D, K) but strictly less capacity. Without loss of generality, we assume that (D, K) is symmetric with respect to the positive semi-axis.

Let $a = \inf\{D \cap \mathbb{R}\}$ and $b = \sup\{D \cap \mathbb{R}\}$, and note that $|a| \leq b$. Note that $a > -\infty$. Let ϕ_c be the elliptic isometry

$$\phi_c(z) = \frac{z - c}{1 + cz},$$

where c is the positive real number with $\mathrm{d}_e[a, c] = \mathrm{d}_e[b, c]$. Looking at the image of $\mathbb{R}$ through the projection defined by relation (1.2), we have that it is the unit circle lying on the plane $\{(x, y, z) \in \mathbb{R}^3 : y = 0\}$. Then c is precisely the pre-image of the point at which the perpendicular bisector of a and b intersects with the right half of that circle. Since $|a| < b$ the antipodal point of c, $-1/c$, is a negative number such that $-1/c < a$. So $-1/c \notin D$ and $\infty \notin \phi_c(D)$. From relation $\mathrm{d}_e[a, c] = \mathrm{d}_e[b, c]$ and (1.4) we also get $|\phi_c(a)| = |\phi_c(b)|$.

Since $(\phi_c(D), \phi_c(K))$ has the same capacity and elliptic area as (D, K) we have

$$\operatorname{cap}(\phi_c(D), \phi_c(K)) = \operatorname{cap}(D, K) = \operatorname{cap}(D^\circ, K^\circ) = \operatorname{cap}(\phi_c(D)^\circ, \phi_c(K)^\circ).$$

So $(\phi_c(D), \phi_c(K))$ must also be circularly symmetric with respect to some semi-axis emanating from 0. By Lemma 2.3, it must be symmetric with respect to either the positive or the negative semi-axis. Without loss of generality, we assume symmetry with respect to the positive semi-axis. Since $-1/c \notin D$, it is easy to see that ϕ_c maps $[a, b]$ to the real line increasingly. It follows that $\phi_c(a) = \inf(\phi_c(D) \cap \mathbb{R})$ and $\phi_c(b) = \sup(\phi_c(D) \cap \mathbb{R})$. This means the entire circle $\{z : |z| = |\phi_c(a)|\}$ is contained in $\overline{\phi_c(D)}$. In addition, if a point d outside this circle belongs to $\phi_c(D)$, then $|d| \in \phi_c(D)$ but $|d| > |\phi_c(a)| = |\phi_c(b)|$, contradiction. So we have shown that, for $r = |\phi_c(a)|$,

$$\{|z| = r\} \subset \overline{\phi_c(D)} \text{ and } \{|z| > r\} \cap \phi_c(D) = \emptyset. \tag{3.3}$$

Since $\operatorname{cap}(\phi_c(D), \phi_c(K)) = \operatorname{cap}(\phi_c(D)^\circ, \phi_c(K)^\circ)$, the condenser $(\phi_c(D), \phi_c(K))$ must also remain circularly symmetric if we apply an elliptic isometry to it. Note that because of (3.3) we have

$$\sup\{\phi_c(D) \cap \mathbb{R}\} = r \text{ and } \inf\{\phi_c(D) \cap \mathbb{R}\} = -r.$$

If we apply ϕ_ϵ to $\phi_c(D)$ for a sufficiently small $\epsilon > 0$ we will have that the resulting condenser must now be symmetric with respect to the negative semi-axis since

$$|\phi_\epsilon(-r)| > |\phi_\epsilon(r)|.$$

This implies that both $\phi_c(D)$ and $\phi_c(K)$ are disks centered at the origin. If that was not the case, we would have a subset A of $(-r, r)$ not in $\phi_c(D)$. If $[a, b]$ is the smallest interval containing A, we would have $|a| \geq |b|$. Applying ϕ_ϵ would change the axis of symmetry but we would still have $|\phi_\epsilon(a)| > |\phi_\epsilon(b)|$ which is in contradiction with symmetry with respect to the negative semi-axis. The same argument can prove that $\phi_c(K)$ must also be a disk centered at the origin.

The converse is obvious since elliptic isometries are conformal maps and do not change condenser capacity. □

4. Proof of Theorem 1.3

We proceed to the proof of the theorem which follows closely the one presented in [2] for the hyperbolic-area-radius case.

(i) Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic map and let $a \in \Omega$. Let $\rho = \sup\{R_e\varDelta(a, r) : \varDelta(a, r) \subset \Omega\}$. Let $0 < r < s < \rho$. Then by (1.8), (1) and (1.5) we have

$$\begin{aligned} 2\pi\left(\log\frac{s}{r}\right)^{-1} &= \operatorname{cap}(\varDelta(a, s), \overline{\varDelta(a, r)}) \\ &\geq \operatorname{cap}(f(\varDelta(a, s)), f(\overline{\varDelta(a, r)})) \\ &\geq \operatorname{cap}(f(\varDelta(a, s))^\circ, f(\overline{\varDelta(a, r)})^\circ) \\ &= 2\pi\left(\log\frac{R_e f(\varDelta(a, s))}{R_e f(\varDelta(a, r))}\right)^{-1}. \end{aligned} \tag{4.1}$$

From the above it follows that

$$\frac{R_e f(\Delta(a,r))}{r} \leq \frac{R_e f(\Delta(a,s))}{s}$$

and so the function Π is increasing.

(ii) If there exist $0 < r < s < \rho$ such that $\Pi(r) = \Pi(s)$, then both inequalities in (4.1) become equalities. That means that f is univalent [12] and we have equality in Theorem 1.2. So there exists an elliptic isometry ϕ_b such that

$$\phi_b(f(\Delta(a,s))) = \Delta(0,s') \text{ and } \phi_b(f(\Delta(a,r))) = \Delta(0,r').$$

From (1.9) we have that

$$\phi_{-a}(\Delta(0,s)) = \Delta(a,s) \text{ and } \phi_{-a}(\Delta(0,r)) = \Delta(a,r).$$

So the function $F(z) = \phi_b \circ f \circ \phi_{-a}(z)$ maps $\Delta(0,s)$ to $\Delta(0,s')$ and $\Delta(0,r)$ to $\Delta(0,r')$. From Lemma 2.4, $F(z) = \lambda z$, and as a result

$$f = \phi_{-b} \circ \lambda z \circ \phi_a.$$

The converse is obvious from the conformal invariance of capacity.

(iii) We start with the observation that

$$\lim_{r\to 0^+} \Pi(r)^2 = \lim_{r\to 0^+} \frac{A_e f(\Delta(a,r))}{A_e \Delta(a,r)}.$$

This follows immediately from

$$A_e f(\Delta(a,r)) = A_e \Delta(0, R_e(\Delta(a,R))) = \frac{\pi\left(R_e(\Delta(a,R))\right)^2}{1 + \left(R_e(\Delta(a,R))\right)^2}$$

and

$$A_e \Delta(a,r) = A_e \Delta(0,r) = \frac{\pi r^2}{1+r^2}.$$

Assuming first that $a = 0$ and using the Lebesque differentiation theorem we have

$$\begin{aligned}
\lim_{r\to 0^+} \Pi(r)^2 &= \lim_{r\to 0^+} \frac{A_e f(\Delta(a,r))}{A_e \Delta(0,r)} \\
&= \lim_{r\to 0} \frac{1+r^2}{\pi r^2} \int_{f(\Delta(0,r))} \frac{1}{(1+|w|^2)^2} A(dw) \\
&\leq \lim_{r\to 0} \frac{1}{\pi r^2} \int_{\Delta(0,r)} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} A(dz) \\
&= \frac{|f'(0)|^2}{(1+|f(0)|^2)^2},
\end{aligned}$$

the inequality arising because f is not necessarily injective.

Finally composing f with a suitable elliptic isometry we remove the restriction $a = 0$.

$$\begin{aligned}\lim_{r\to 0^+} \Pi(r)^2 &= \lim_{r\to 0^+} \frac{A_e f(\Delta(a,r)}{A_e \Delta(a,r)} \\ &= \lim_{r\to 0^+} \frac{A_e f\circ\phi_{-a}\,(\Delta(a,r))}{A_e \Delta(a,r)} \\ &= \frac{|(f\circ\phi_{-a}(0))'|^2}{(1+|f\circ\phi_{-a}(0)|^2)^2} \\ &= \frac{|f'(a)|^2(1+|a|^2)^2}{(1+|f(a)|^2)^2}.\end{aligned}$$

(iv) From (iii) and (i) it follows that

$$\begin{aligned}\frac{|f'(a)|^2(1+|a|^2)}{1+|f(a)|^2} &= \lim_{r\to 0^+}\Pi(r) = \lim_{r\to 0^+}\frac{R_e f(\Delta(a,r)}{r} \\ &\le \frac{R_e f(\Delta(a,\rho))}{\rho} = \frac{1}{\rho}\left(\frac{A_e f(\Delta(a,\rho))}{\pi - A_e f(\Delta(a,\rho))}\right)^{\frac{1}{2}}.\end{aligned}$$

□

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